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On the persistent homology of almost surely C^0 stochastic processes

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Abstract

The persistent homology of random processes has been a question of interest in the TDA community. This paper aims to extend a result of Chazal and Divol and to reconcile the probabilistic approach to the study of the topology of superlevel sets of random processes in dimension 1. We provide explicit descriptions of the barcode for Brownian motion and the Brownian bridge. Additionally, we use these results on almost surely C^0-processes to describe the barcodes of sequences which admit these objects as limits, up to some quantifiable error.

1 Introduction

One of the questions of interest in the theory of persistent homology is the following: given a random function on some topological space X, what can we say about the barcode $\mathcal{B}(X)$ of this process? The study of the topology of (super)level-sets of random functions has been a subject of interest in probability theory for a long time. Many advances in this direction have been provided by a myriad of authors [1, 2, 20, 25, 29, 34]. Most prominently for this paper, by Le Gall and Duquesne, who gave a construction of a tree from any continuous function $f : [0, 1] \to \mathbb{R}$ [13], and who interpreted different properties of these trees to give fine results about Lévy processes [14]. Picard later linked the upper-box dimension of these trees to the regularity of the function f [33]. In essence, these trees have proved to be a fruitful and natural setting from which many results regarding

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the topology of the superlevel sets of the function f stem [10, 11, 15, 30, 32]. A natural question is whether, or indeed how, these results are applicable to the persistent homology of stochastic processes. The answer turns out to be total: the study of barcodes and trees are completely equivalent, at least in degree 0 of homology. This has been established in [32], in which a dictionary between H_0-barcodes and the probabilist's trees was constructed.

Parallel to these developments, some results regarding the persistent homology of Brownian motion have also been provided by the topological data analysis (TDA) community. In particular, Chazal and Divol gave a formula for the distribution of the number of points $N^{x,y}$ lying inside a given rectangle $]-\infty, x] \times [y, \infty[$ in the persistence diagram of Brownian motion, $\mathrm{Dgm}(B)$ [8]. Similar results were obtained by Baryshnikov [3], who computed exactly $\mathbb{E}[N^{x,y}]$ for the Brownian motion with a linear drift over the ray $[0, \infty[$. The goal of this paper is threefold and will be tackled in

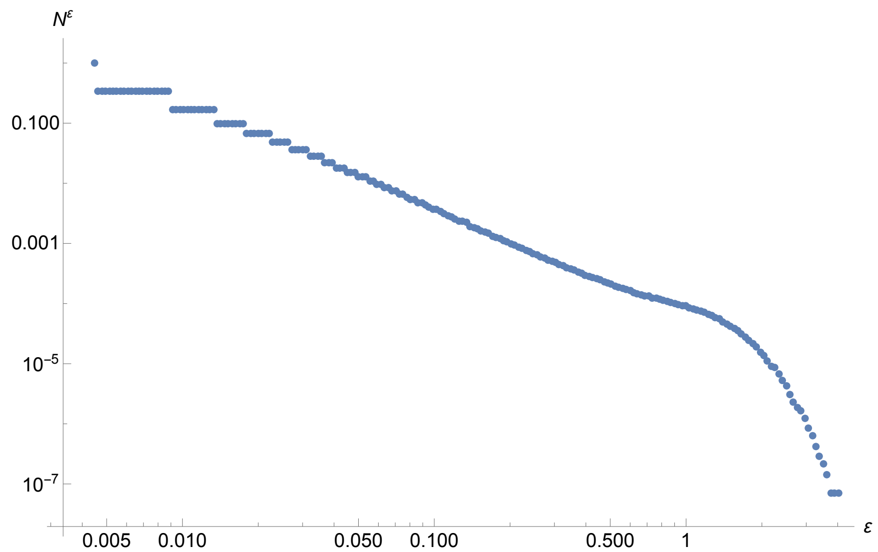

Figure 1: The expected number of bars of length $\geq \varepsilon$, $\mathbb{E}[N^\varepsilon]$, as a function of ε as calculated with simulations of Brownian paths by random Rademacher walks.

the following order. First, we apply and reinterpret the results obtained in the probability literature in the context of the persistent homology of a stochastic process X. A comfortable setting to do this is to suppose that the stochastic process satisfies the strong Markov property. Along the way, we give asymptotics to Chazal and Divol's formula and reinterpret these results in terms of the local time of Brownian motion. Second, we study the results obtained in this first section quantitatively, by studying two specific Markov processes: Brownian motion and the Brownian bridge. This will allow us to make quantitative statements about simulations performed for both of these processes, which have been depicted in figure 1. Third, establish, by virtue of example, the paradigm that these (in general) C^0-objects can be seen as limiting cases of sequences of interest, but which might be harder to study than the limiting object. We will illustrate this by considering sequences of random trigonometric polynomials, which admit an almost sure C^0 limit, and deduce from the barcodes of these limits elements about the sequences in question.

In this paper, we will mostly be concerned with almost surely C^0 processes, but it is noteworthy that the study of smooth random fields (and their topology) is currently an open field of research in probability theory. A good introduction to the smooth setting is provided by the celebrated books of Adler and Taylor [1], and that of Azaïs and Wschebor [2].

Our contribution can be summarized along the five following points.

1. We give a link between $N^{x,y}$ and the number of bars N^ε of length $\geq \varepsilon$ (section 2.5.2). We also give the asymptotics of Chazal and Divol's formula in the $x \to y$ regime and give an interpretation of this result in terms of the local time of Brownian motion (proposition 2.3). Furthermore, we also compare our results to those of Baryshnikov (proposition 2.20);

2. A general formula for the distribution of the functional N^ε for any Markov process (theorem 2.8). We further give useful inequalities (theorem 2.10) on the formulæ obtained and study two asymptotic regimes, $\varepsilon \to 0$ (for which the results were already known in the literature [33]) and $\varepsilon \to \infty$ (theorem 2.10).

3. The explicit calculation of the distribution of N^ε for the Brownian motion (proposition 2.14), as well as the asymptotic study of the formulæ obtained in the two regimes described above, in particular, we determine that the distribution of $N^{x,x+\varepsilon}$ and N^ε are analytic in x and ε for Brownian motion (theorem 2.17 and proposition 2.18) and that $\mathbb{E}[N^{x,x+\varepsilon}]$ and $\mathbb{E}[N^\varepsilon]$ are as well. We also give the asymptotic behaviour of N^ε for the Brownian bridge (proposition 2.22);

4. We explore the consequences of L^∞-stability in terms of inequalities on the functionals N^ε of a limiting process X and a convergent sequence X_n such that $X_n \xrightarrow[\text{a.s.}]{L^\infty} X$, in particular, we find that $N^\varepsilon_{X_n} \xrightarrow{L^1} N^\varepsilon_X$ (theorem 3.1);

5. Since this last result requires a quantitative estimate of the rate of convergence of X_n to be useful in practice, we complement this result with statements quantifying the almost sure rates of convergence of random series (theorem 3.8). We apply these results to give some

estimates regarding the barcodes of some models of random trigonometric polynomials (propositions 3.12, 3.13 and 3.15), random walks and empirical processes (proposition 3.19).

2 Markov processes on $[0, 1]$

For the rest of this section, let X be a stochastic process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with values in $C^0([0,1], \mathbb{R})$. We will also assume – unless otherwise specified – that X satisfies the strong Markov property. For commodity, we will make a distinction between Markov processes which are *periodic in time* (we will also say that they are defined on $\mathbb{S}_1$) and those which are not. By *periodic in time*, we mean processes X which almost surely achieve their infimum at 0 and for which $X_0 = X_1$ almost surely.

2.1 Chazal and Divol's and Baryshnikov's formulæ

Chazal and Divol [8] were able to describe precisely the number of points $N^{x,y}$ in the persistence diagram of Brownian motion lying inside a rectangle $]-\infty, x] \times [y, \infty[$ of $\mathrm{Dgm}(B)$.

Proposition 2.1 (Chazal, Divol, [8])**.** For $0 < x < y$, the distribution of $N^{x,y}$ is

$$\mathbb{P}(N^{x,y} \geq k) = \int_{\Sigma_{2k-1}} \mathrm{d}^{k-1}\mathbf{t}\ \mathrm{d}^{k-2}\mathbf{s}\ \ \psi(x, t_1)\psi(y-x, s_1)\psi(y-x, t_2)\cdots\psi(y-x, t_k)$$

where Σ_{2k-1} denotes the corner of the domain bounded by the $(2k-1)$-simplex and we note a vector in $\mathbb{R}^{2k-1}$ by $(t_1, s_1, \cdots, s_{k-2}, t_{k-1})$ and

$$\psi(x,t) := \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}} \ .$$

In particular, $\mathbb{E}[N^{x,y}]$ is finite for $x < y$ and equal to

$$\mathbb{E}[N^{x,y}] = \sum_{k \geq 1} \mathbb{P}(N^{x,y} \geq k)$$

Remark 2.2. An analogous expression holds for $x < 0$ by the reflection principle.

Despite being explicit, this fact is difficult to use. Nonetheless, using this result Chazal and Divol established that $\mathbb{E}[N^{x,y}]$ was C^1 in x and y. Given this formula alone, it is difficult to give asymptotics of this formula, but from a probabilistic standpoint, the asymptotics become obvious when $x \to y$. Indeed, in this limit and up to an appropriate renormalization $N^{x,y}$ is the local time at y.

Proposition 2.3. Denote $\varphi(x,t)$ the probability density function of a $\mathcal{N}(0,t)$ random variable, then

$$\mathbb{E}[N^{x,y}] \sim \frac{1}{2\,|x-y|} \int_0^1 \varphi(y,s)\; ds \quad \text{as } x \to y$$

Proof. Let $N_0^\varepsilon(T)$ denote the number of down-crossings of a reflected Brownian motion from $\varepsilon > 0$ to 0 before time T. It is known (Itô, [19, §2.4]) that for $[0,T]$

$$2\varepsilon N_0^\varepsilon(T) \xrightarrow[\varepsilon\to 0]{\text{a.s}} L_0(T)$$

where $L_0(T)$ denotes the local time at 0 of the Brownian motion. Furthermore, we have an almost sure asymptotic dependence

$$N_0^\varepsilon(T) \sim \frac{L_0(T)}{2\varepsilon} \quad \text{as } \varepsilon \to 0$$

The expected value of the local time of Brownian motion is well-known so applying this result to $|B_t - y|$, we have the statement of the proposition. ■

This result reflects the importance of insights stemming from probability theory when studying asymptotics of formulæ describing $\mathcal{B}(B)$. This will also be the case in what will follow.

For Brownian motion with a drift $B_t^\mu := B_t + \mu t$ on the ray $[0,\infty[$, a similar statement was obtained by Baryshnikov for $\mu > 0$ [3].

Proposition 2.4 (Baryshnikov, [3])**.** The expected value of $N_{B^\mu}^{x,x+\varepsilon}$ for $x > 0$ is given by

$$\mathbb{E}\Big[N_{B^\mu}^{x,x+\varepsilon}\Big] = \frac{1}{e^{2\mu\varepsilon} - 1}\,.$$

In particular, as $\varepsilon \to 0$, the following asymptotic relation holds

$$\mathbb{E}\Big[N_{B^\mu}^{x,x+\varepsilon}\Big] \sim \frac{1}{2\mu\varepsilon} - \frac{1}{2} + \frac{1}{6}\mu\varepsilon + O(\mu^3\varepsilon^3) \quad \text{as } \varepsilon \to 0\,.$$

Notice that we expect this result to diverge as $\mu \to 0$, since Brownian motion almost surely crosses $[x, x+\varepsilon]$ infinitely many times. On the other hand, the result is ensured to be finite for $\mu > 0$, since Brownian motion is almost surely $(\frac{1}{2} - \delta)$-Hölder continuous for all $\delta > 0$. In particular, looking at this process on the ray $[0,\infty[$ renders this expectation completely independent of x.

2.2 The number of bars of length $\geq \varepsilon$

The previous formula provides us with some insight on the allure of the barcode locally. However, it lacks a holistic view of the barcode of B, as it is inherently limited to rectangles. For the rest of this paper, we will focus on studying a functional supplementary to $N^{x,y}$: the number of bars of length $\geq \varepsilon$, which we will denote N^ε. As we will see, N^ε is best understood in the context of trees.

Let us briefly recall the construction of a tree from a continuous function $f : [0,1] \to \mathbb{R}$.

Definition/Proposition 2.5. Let $x < y \in [0,1]$, then the function

$$d_f(x,y) := f(x) + f(y) - 2 \min_{t \in [x,y]} f(t)$$

is a pseudo-distance on $[0,1]$ and the quotient metric space

$$T_f := [0,1]/\{d_f = 0\}$$

with distance d_f is a rooted $\mathbb{R}$-tree, whose root coincides with the image in T_f of the point in $[0,1]$ at which f achieves its infimum.

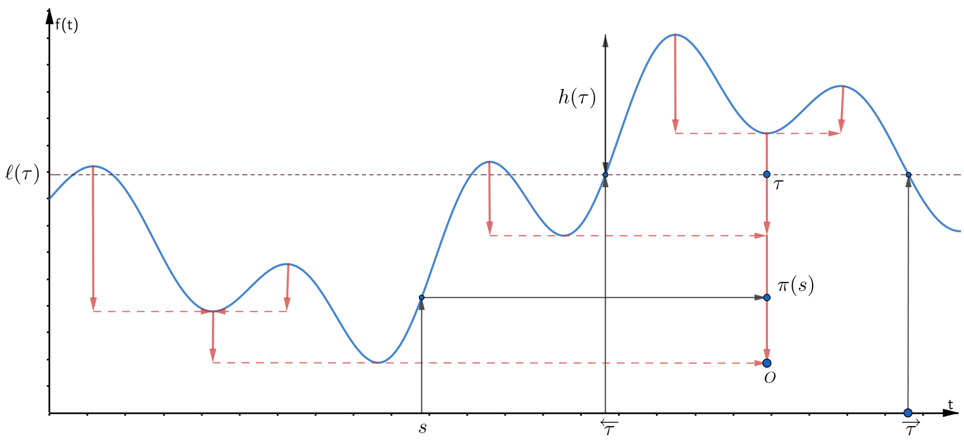

Figure 2: A function f and its associated tree T_f in red.

The tree T_f has the particularity that its branches correspond to connected components of the superlevel sets of f, as illustrated by figure 2. To define N^ε on this tree, it is first necessary to introduce the so-called ε-simplified or ε-trimmed tree of T_f^ε. This object is obtained by "giving a haircut" of length ε to T_f. More precisely, if we define a function $h : T_f \to \mathbb{R}$ which to a point $\tau \in T_f$ associates the distance from τ to the highest leaf above τ with respect to the filtration on T_f induced by f, then

Definition 2.6. Let $\varepsilon \geq 0$. An **ε-trimming** or **ε-simplification** of T_f is the metric subspace of T_f defined by

$$T_f^\varepsilon := \{\tau \in T_f \,|\, h(\tau) \geq \varepsilon\}$$

With this definition, we can interpret N^ε geometrically as being equal to the number of leaves of T_f^ε. To see why, it is helpful to also recall the correspondence between trees and barcodes. The idea is that, starting from T_f, we can look at the longest branch (starting from the root) of T_f. This branch corresponds to the longest bar of $\mathcal{B}(f)$, since branches of T_f correspond to connected components of the superlevel sets of f. Next, we erase this longest branch and, on the remaning (rooted) forest, look for the next longest branch. This will be the second longest bar of the barcode. Proceeding iteratively in this way, we retrieve $\mathcal{B}(f)$. An illustration of this can be found in figure 3.

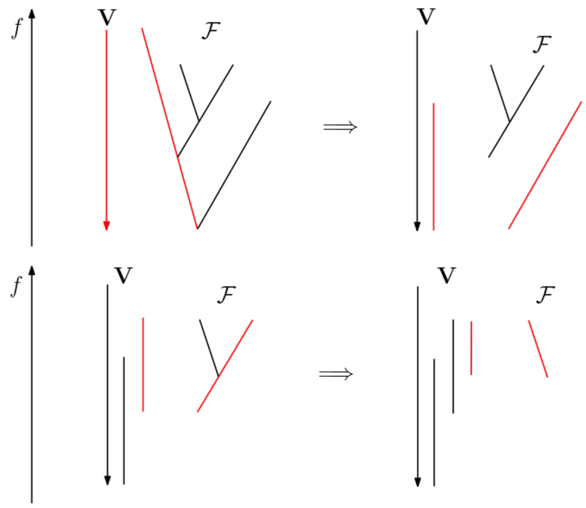

Figure 3: A depiction of the first steps of the algorithm which assigns a barcode $\mathcal{B}(f)$ to a tree T_f.

Of course, we could also count N^ε by counting the number of times we go up by at least ε from a local minimum and down by at least ε from a local maximum. This idea can be formalized by the following sequence, originally introduced by Neveu *et al.* [30].

Definition 2.7. Setting $S_0^\varepsilon = T_0^\varepsilon = 0$, we define a sequence of times by induction

$$T_{i+1}^\varepsilon := \inf\left\{ t \geq S_i \,\middle|\, \sup_{[S_i^\varepsilon, t]} f - f(t) > \varepsilon \right\}$$

$$S_{i+1}^\varepsilon := \inf\left\{ t \geq T_{i+1} \,\middle|\, f(t) - \inf_{[T_{i+1}^\varepsilon, t]} f > \varepsilon \right\}$$

Counting the number of bars of length ε is thus exactly to count the number of up and downs we make. More precisely

$$N^\varepsilon = \inf\{i \,|\, T_i^\varepsilon \text{ or } S_i^\varepsilon = \inf \emptyset\} \tag{2.1}$$

by which we mean that it is the smallest i such that the set over which T_i^ε or S_i^ε are defined as infima is empty. If we apply these definitions to the stochastic process X, this sequence is in fact a sequence of stopping times. This remark gives us a theorem *à la* Chazal-Divol for the distribution of N^ε.

Theorem 2.8. *Let X be a continuous stochastic process on M where M is $[0,1]$ or $\mathbb{S}_1$ and suppose that X satisfies the strong Markov property. Then*

$$\mathbb{P}(N^\varepsilon \geq k) = \begin{cases} \displaystyle\int_{\Sigma_{2k-2}} d^{k-1}\mathbf{t}\, d^{k-1}\mathbf{s}\ \ \langle s_{k-1}|t_{k-1}\rangle\langle t_{k-1}|s_{k-2}\rangle \cdots \langle t_1|0\rangle & \text{if } M = [0,1] \\ \displaystyle\int_{\Sigma_{2k-1}} d^{k}\mathbf{t}\, d^{k-1}\mathbf{s}\ \ \langle t_k|s_{k-1}\rangle\langle s_{k-1}|t_{k-2}\rangle \cdots \langle t_1|0\rangle & \text{if } M = \mathbb{S}_1 \end{cases}$$

for $k > 1$, where $\langle t_1|0\rangle = \mathbb{P}(T_1 = t_1|T_0 = 0) = \mathbb{P}(T_1 = t_1)$,

$$\langle s_k|t_i\rangle = \mathbb{P}(S_k = s_k \,|\, T_i = t_i)$$

and where

$$\Sigma_{2k-2} := \left\{(t_1, s_1, t_2, \cdots, s_{k-1}) \in \mathbb{R}^n \,\middle|\, 0 \leq t_1 \leq s_1 \leq t_2 \leq \cdots \leq s_{k-1} \leq 1\right\}$$
$$\Sigma_{2k-1} := \left\{(t_1, s_1, t_2, \cdots, t_k) \in \mathbb{R}^n \,\middle|\, 0 \leq t_1 \leq s_1 \leq t_2 \leq \cdots \leq t_k \leq 1\right\}$$

Finally, $\mathbb{P}(N^\varepsilon \geq 1) = \mathbb{P}(\sup_M X - \inf_M X \geq \varepsilon)$.

Notation 2.9. The range of the process X will play a considerable role in what will follow. Let us denote the range R of X by

$$R := \sup_{[0,1]} X - \inf_{[0,1]} X$$

Proof of theorem 2.8. First, we note that $N^\varepsilon \geq 1$ if and only if $R \geq \varepsilon$. It follows that

$$\mathbb{P}(N^\varepsilon \geq 1) = \mathbb{P}(R \geq \varepsilon)$$

Having treated the case $k = 1$, suppose that $k > 1$. It is necessary to consider two separate cases

1. If the process is **not** periodic in time, then $N^\varepsilon \geq k$ if and only if $S_{k-1}^\varepsilon \leq 1$. Note that the direct implication is trivial, by the counting formula for N^ε given in equation 2.1. Conversely, if $S_{k-1}^\varepsilon \leq 1$, then we know that we have generated $k-1$ bars of length $\geq \varepsilon$ on the interval $[0, T_{k-1}^\varepsilon]$, but since $S_{k-1}^\varepsilon \leq 1$, there is at least one supplementary bar of length $\geq \varepsilon$ generated by the boundary at 1. Figure 4 illustrates this is in more detail.

2. If the process is periodic in time (or defined on $\mathbb{S}_1$), then by our convention it almost surely attains its infimum at 0 and $X_1 = X_0$. Then, $N^\varepsilon \geq k$ if and only if $T_k^\varepsilon \leq 1$. Here, the converse implication is clear

by equation 2.1, but the direct one is not. Suppose then that there k bars in the barcode of X. Since the process attains its infimum at 0 and at 1, the times $T_i^\varepsilon \leq 1$ are in bijection with the maxima of depth at least ε. It follows that if there are k bars, there must be k such maxima, so $T_k^\varepsilon \leq 1$.

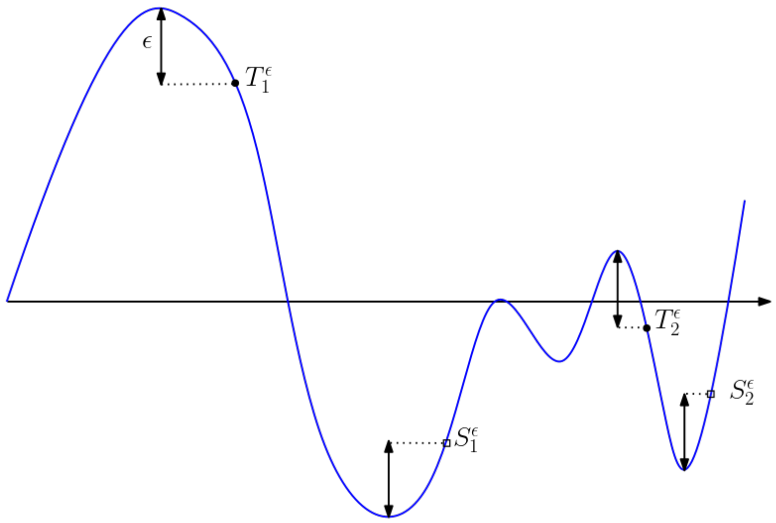

Figure 4: A function f in blue along with the times T_i^ε and S_i^ε indicated. Because of the boundary this function has exactly 3 bars of length $\geq \varepsilon$ and not just 2.

With these equivalences and with the remark that the times T_i^ε and S_i^ε are stopping times, an application of the Chapman-Kolmogorov equation shows that the probabilities $\mathbb{P}(N^\varepsilon \geq k)$ can be written exactly as in the statement of the theorem, since X has the strong Markov property. ∎

Just as before, this formula is conceptually useful, but impractical. Nevertheless, it admits relatively simple asymptotic expansions in the regimes where $\varepsilon \to 0$ and $\varepsilon \to \infty$. Just as in the Chazal-Divol case, the asymptotics as $\varepsilon \to 0$ are be founded on geometrical and probabilistic arguments.

2.3 The $\varepsilon \to \infty$ regime

If ε is very large, it is reasonable to expect that the event where there are lots of large bars in $\mathcal{B}(X)$ is rare. Since X is assumed to be Markov, it is not too hard to imagine that $\mathbb{P}(N^\varepsilon \geq k)$ should be dominated by the probability of looking at roughly k independent copies of X and examining the probability that each of these copies contains at least one variation of at least ε. That this should be the case is clear, as with k copies of X, the process has "more time" to generate large bars, as in the first case.

This intuition turns out to be correct and is corroborated by the following theorem.

Theorem 2.10. *For all $\varepsilon > 0$, the following inequalities are satisfied*

$$\mathbb{P}(N^\varepsilon \geq k) \leq \begin{cases} \mathbb{P}(R \geq \varepsilon)^{2k-2} & \text{if } M = [0,1] \\ \mathbb{P}(R \geq \varepsilon)^{2k-1} & \text{if } \Omega = \mathbb{S}_1 \end{cases}$$

$$\mathbb{E}[N^\varepsilon] \leq \begin{cases} \dfrac{\mathbb{P}(R \geq \varepsilon)[1 + \mathbb{P}(R \geq \varepsilon) - \mathbb{P}(R \geq \varepsilon)^2]}{1 - \mathbb{P}(R \geq \varepsilon)^2} & \text{if } \Omega = [0,1] \\ \dfrac{\mathbb{P}(R \geq \varepsilon)}{1 - \mathbb{P}(R \geq \varepsilon)^2} & \text{if } M = \mathbb{S}_1 \end{cases}$$

In particular,the following asymptotic relation holds

$$\mathbb{E}[N^\varepsilon] \sim \mathbb{P}(R \geq \varepsilon) \quad \text{as } \varepsilon \to \infty\,.$$

Proof. Let us prove the statement for $M = \mathbb{S}_1$. Notice that

$$\mathbb{P}\left(\sup_{\tau \in [s,t]} \left[\sup_{[s,\tau]} X - X_\tau\right] \geq \varepsilon\right) \leq \mathbb{P}\left(\sup_{[0,1]} \left[\sup_{[0,1]} X - X_t\right] \geq \varepsilon\right)$$

The supremum on the right hand side is really nothing other than R since the supremum of the difference between the running supremum and the process is achieved at the end of the interval. Thus,

$$\mathbb{P}\left(\sup_{\tau \in [s,t]} \left[\sup_{[s,\tau]} X - X_\tau\right] \geq \varepsilon\right) \leq \mathbb{P}(R \geq \varepsilon)\,. \tag{2.2}$$

The probability $\mathbb{P}(N^\varepsilon \geq k)$ can be written in terms of the stopping times T_i^ε and S_i^ε and their increments by the (strong) Markov property of X as prescribed by theorem 2.8. Recall that

$$\mathbb{P}(N^\varepsilon \geq k) = \int_{\Sigma_{2k-1}} \mathrm{d}^k \mathbf{t}\, \mathrm{d}^{k-1}\mathbf{s}\;\; \langle 0|t_1\rangle\langle t_1|s_1\rangle \cdots \langle s_{k-1}|t_k\rangle$$

where Σ_{2k-1} denotes the simplex

$$\Sigma_{2k-1} := \left\{(t_1, s_1, \cdots, s_{k-1}, t_k) \in \mathbb{R}^{2k-1} \;\middle|\; 0 \leq t_1 \leq s_2 \leq \cdots \leq s_{k-1} \leq t_k \leq 1\right\}$$

By the definition of these stopping times we know that

$$\langle s \leq T_i^\varepsilon \leq t | S_{i-1} = s\rangle = \mathbb{P}\left(\sup_{\tau \in [s,t]} \left[\sup_{[s,\tau]} X - X_\tau\right] \geq \varepsilon\right)$$

$$\langle t \leq S_i^\varepsilon \leq s | T_i = t\rangle = \mathbb{P}\left(\sup_{\tau \in [t,s]} \left[X_\tau - \inf_{[t,\tau]} X\right] \geq \varepsilon\right)$$

Both of these expressions are $\leq \mathbb{P}(R \geq \varepsilon)$. To simplify the notation let $|T_i^\varepsilon = t_1\rangle = |t_1\rangle$ and $|S_i^\varepsilon = s_1\rangle = |s_1\rangle$. Integrating over the variable t_k between s_{k-1} and 1

$$\begin{aligned}\mathbb{P}(N^\varepsilon \geq k) &= \int_{\Sigma_{2k-2}} \mathrm{d}^{k-1}\mathbf{t}\, \mathrm{d}^{k-1}\mathbf{s} \;\; \langle s_{k-1} \leq T_k^\varepsilon \leq 1 | s_{k-1}\rangle \cdots \langle s_1|t_1\rangle\langle t_1|0\rangle \\ &\leq \mathbb{P}(R \geq \varepsilon) \int_{\Sigma_{2k-2}} \mathrm{d}^{k-1}\mathbf{t}\, \mathrm{d}^{k-1}\mathbf{s} \;\; \langle s_{k-1}|t_{k-1}\rangle \cdots \langle s_1|t_1\rangle\langle t_1|0\rangle\end{aligned}$$

Carrying out the subsequent $2k-2$ integrations and by repeated use of the inequality given in equation 2.2, we finally obtain that

$$\mathbb{P}(N^\varepsilon \geq k) \leq \mathbb{P}(R \geq \varepsilon)^{2k-1} .$$

For $\varepsilon \to \infty$, the probability $\mathbb{P}(R \geq \varepsilon) < 1$ so the above condition guarantees the summability of the series $\mathbb{E}[N^\varepsilon]$ as soon as $\varepsilon > 0$. We have a bound

$$\begin{aligned}\mathbb{E}[N^\varepsilon] &= \sum_{k=1}^{\infty} \mathbb{P}(N^\varepsilon \geq k) \\ &\leq \frac{\mathbb{P}(R \geq \varepsilon)}{1 - \mathbb{P}(R \geq \varepsilon)^2}\end{aligned}$$

The asymptotic relation in the statement of the proposition follows clearly from this fact, as $\mathbb{P}(R \geq \varepsilon) \to 0$ as $\varepsilon \to \infty$. Since the first term of $\mathbb{E}[N^\varepsilon]$ is $\mathbb{P}(R \geq \varepsilon)$ and given the inequality on $\mathbb{E}[N^\varepsilon]$ above, the asymptotic relation must hold since the subsequent terms of the sum are suppressed.

Finally, if $M = [0,1]$ the persistent homology computed on the interval is the homology as computed for manifolds with boundary, this has the effect that a kth bar can occur as soon as $S_{k-1}^\varepsilon \leq 1$, whose probability can be written as:

$$\mathbb{P}(N^\varepsilon \geq k) = \int_{\Sigma_{2k-2}} \mathrm{d}^{k-1}\mathbf{t}\, \mathrm{d}^{k-1}\mathbf{s} \;\; \langle s_{k-1}|t_{k-1}\rangle \cdots \langle s_1|t_1\rangle\langle t_1|0\rangle$$

which implies that we only carry out $2k-2$ integrations, hence the revised estimate. As before, the bound for $\mathbb{E}[N^\varepsilon]$ is obtained by summation of the series of bounds of $\mathbb{P}(N^\varepsilon \geq k)$. ∎

The proof of this theorem has the following interesting consequence.

Proposition 2.11. If $\mathbb{P}(N^\varepsilon \geq k)$ are continuous in ε and for all $\varepsilon > 0$ $\mathbb{P}(R \geq \varepsilon) < 1$, then $\mathbb{E}[N^\varepsilon]$ is continuous.

Proof. We will show the result for non-periodic processes $[0,1]$. The result for periodic processes follows analogously, with slightly modified estimates.

It suffices to show that the series $\sum_{k=n}^{\infty} \mathbb{P}(N^\varepsilon \geq k)$ converges uniformly (and absolutely). We have the inequality

$$\sum_{k=n}^{\infty} \mathbb{P}(N^\varepsilon \ \geq k) \leq \frac{\mathbb{P}(R \geq \varepsilon)^{2n-2}}{1 - \mathbb{P}(R \geq \varepsilon)^2} .$$

But $\mathbb{P}(R \geq \varepsilon) < 1$ as soon as $\varepsilon > 0$, implies that $\mathbb{E}[N^\varepsilon]$ converges uniformly and absolutely on every compact of $]0, \infty[$. We conclude that $\mathbb{E}[N^\varepsilon]$ must also be continuous in ε. ■

2.4 The $\varepsilon \to 0$ regime

Intuitively, if ε is small, the number of bars N^ε should strongly depend on the regularity of the process, as ultimately N^ε counts the number of "oscillations" of size ε. In a very precise sense, regularity almost fully determines the asymptotics of N^ε in the $\varepsilon \to 0$ regime, but more is true. With some work, it is possible to link the regularity of the process to metric invariants of T_f and to quantities of interest in the TDA literature, such as the Pers_p functional of [6, 9, 12, 28, 36], hereby denoted ℓ_p defined as

$$\ell_p(f) := \left[\sum_{b \in \mathcal{B}(f)} \ell(b)^p \right]^{1/p} .$$

where $\ell(b)$ denotes the length of the bar b.

Theorem 2.12 (Picard, §3 [33] and [32])**.** *Given a continuous function* $f : [0, 1] \to \mathbb{R}$*,*

$$\mathcal{V}(f) = \mathcal{L}(f) = \overline{\dim}\, T_f = \limsup_{\varepsilon \to 0} \frac{\log N^\varepsilon}{\log(1/\varepsilon)} \vee 1$$

where $\overline{\dim}$ *denotes the upper-box dimension,* $a \vee b = \max\{a, b\}$*,*

$$\mathcal{V}(f) := \inf\{p \,|\, \|f\|_{p-var} < \infty\} \quad \textit{and} \quad \mathcal{L}(f) := \inf\{p \,|\, \ell_p < \infty\} .$$

This theorem gives an almost complete account of the asymptotics of N^ε as $\varepsilon \to 0$ for *any* continuous function, and by extension any almost surely continuous stochastic process X. Adding the supplementary hypothesis that X is Lévy and almost surely nowhere monotone, Picard showed that these asymptotics can be rendered exact.

Theorem 2.13 (Behaviour of Lévy trees, Picard, §3 [33])**.** *Let X be a Lévy process on* $[0, 1]$ *and suppose that, almost surely, X has no interval on which it is monotone. Define:*

$$T^\varepsilon := \inf\left\{ t \,\middle|\, \sup_{[0,t]} X - X_t > \varepsilon \right\} \quad \textit{and} \quad S^\varepsilon := \inf\left\{ t \,\middle|\, X_t - \inf_{[0,t]} X > \varepsilon \right\}$$

and let:

$$\xi(\varepsilon) := \mathbb{E}[S^\varepsilon + T^\varepsilon] \ .$$

Then,

$$\xi(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0 \qquad \textit{and} \qquad \xi(\varepsilon) N^\varepsilon \xrightarrow[\varepsilon \to 0]{\mathbb{P}} 1$$

Furthermore, if $\xi(\varepsilon) = O(\varepsilon^\alpha)$ for some $\alpha > 0$, the convergence of $\xi(\varepsilon)N^\varepsilon$ is almost sure.

A posteriori, a similar result can also be obtained using the work of Le Gall on properties of Lévy trees. Indeed, if the so-called *branching mechanism* of the Lévy process is stable, the Hausdorff dimension and the upper-box dimension of the tree are equal. As remarked in [32], it is then a consequence of an extension of Picard's theorem that the limits in Picard's theorem exist, from which exact asymptotics also follow. The details on Le Gall's construction can be found in [14]. Nonetheless, Picard's version of this result is useful, as it provides us with a way of calculating the constants of the asymptotic expansion, whenever possible.

These two theorems show us that, while the regime $\varepsilon \to 0$ is the hardest to analyze simply by looking at the formula of theorem 2.8, in a sense it is also the simplest, as almost sure asymptotic bounds exist.

2.5 Brownian motion and the Brownian bridge

Any account on C^0 stochastic processes on $[0,1]$ would not be complete without treatment of Brownian motion. Letting B denote a standard Brownian motion, throughout this section we will aim to describe $\mathcal{B}(B)$ and subsequently give an analogous treatment for the Brownian bridge W.

2.5.1 Brownian motion

For Brownian motion, it is possible to improve on the results of theorem 2.10 since we have explicit expressions for $\mathbb{P}(N^\varepsilon \geq k)$.

Proposition 2.14. For $k > 1$, the distribution of N^ε is given by

$$\mathbb{P}(N^\varepsilon \geq k) = \int_{\Sigma_{2k-2}} \mathrm{d}^{k-1}\mathbf{t} \ \mathrm{d}^{k-1}\mathbf{s} \ \ f_T(\varepsilon, t_1) f_T(\varepsilon, s_1) \cdots f_T(\varepsilon, t_{k-1}) f_T(\varepsilon, s_{k-1})$$

where

$$\Sigma_n = \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n \ \middle| \ \sum_{i=1}^n x_i \leq 1 \right\}$$

and

$$f_T(\varepsilon, t) = \sum_{k=-\infty}^{\infty} (-1)^{k-1} \, \psi((2k-1)\varepsilon, t) \, .$$

If $k = 1$, then,

$$\mathbb{P}(N^\varepsilon \geq 1) = \mathbb{P}(R \geq \varepsilon) = 4\sum_{k=1}^{\infty}(-1)^{k-1}k\,\mathrm{erfc}\left[\frac{k\varepsilon}{\sqrt{2}}\right]$$
$$= 1 - 4\sum_{k=-\infty}^{\infty}\frac{e^{-\frac{(2k-1)^2\pi^2}{2\varepsilon^2}}}{\varepsilon^2}\left[1 + \frac{\varepsilon^2}{\pi^2(2k-1)^2}\right]$$

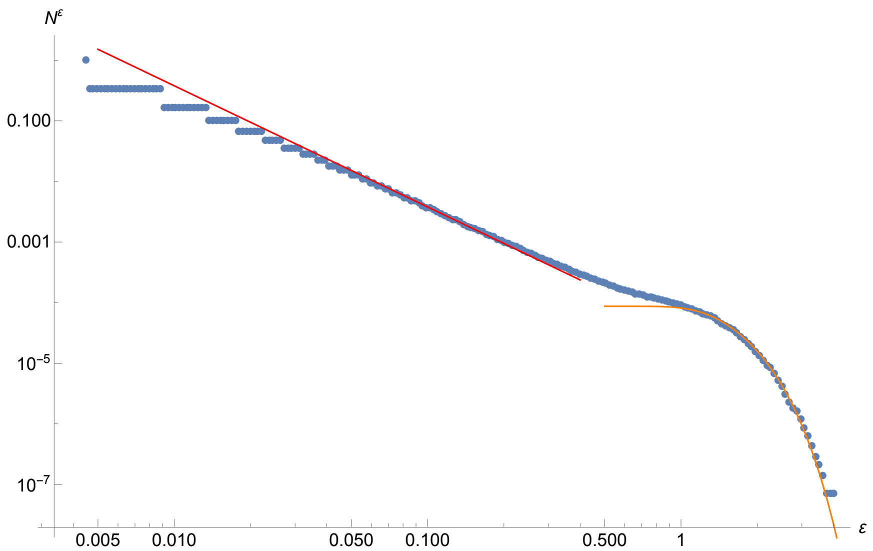

Figure 5: $\mathbb{E}[N^\varepsilon]$ as calculated by simulations of Brownian motion using a random walk. For small ε, we see that the discretization in the points is due to the fact that the random walk's steps were taken to be Bernouilli. In red, we have depicted the asymptotic result as $\varepsilon \to 0$ and in orange, the corresponding result for $\varepsilon \to \infty$. Interestingly, the study of these two regimes almost fully characterizes the curve of $\mathbb{E}[N^\varepsilon]$.

Proof. Brownian motion is a Markov process and has independent increments (in fact, it is a Lévy process). In particular, $T_i^\varepsilon - S_{i-1}^\varepsilon$ and $S_i^\varepsilon - T_i^\varepsilon$ are identically distributed for all i. In distribution, these increments are all equal to the stopping time T^ε defined in the statement of the theorem on the behaviour of Lévy trees (theorem 2.13). For Brownian motion, T^ε is exactly the first hitting time of ε of a reflected Brownian motion.

The distribution f_T of this process has been calculated by Feller to be [16]

$$f_T(\varepsilon, t) := \sum_{k=-\infty}^{\infty}(-1)^{k-1}\psi((2k-1)\varepsilon, t)$$
$$= \sum_{k=-\infty}^{\infty}\frac{(-1)^{k-1}(2k-1)\pi}{4\varepsilon^2}\,e^{-\frac{(2k-1)^2 t}{8\varepsilon^2}}\,.$$

Since we have the distribution of the *increments* of the stopping times T_i^ε and S_i^ε, a reparametrization of the integral of theorem 2.8 is necessary. This reparametrization amounts to looking at the simplex defined as

$$\Sigma_n := \left\{ (t_1, \cdots, t_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} t_i \leq 1 \right\}$$

from which the integral formula in the statement of the theorem follows for $k > 2$. For $k = 1$, it suffices by theorem 2.8 to look at the range of Brownian motion. It is also a classical result by Feller [16] that this range R has distribution

$$\begin{aligned}\mathbb{P}(R \geq \varepsilon) &= 4 \sum_{k=1}^{\infty} (-1)^{k-1} k \operatorname{erfc}\left[\frac{k\varepsilon}{\sqrt{2}}\right] \\ &= 1 - 4 \sum_{k=-\infty}^{\infty} \frac{e^{-\frac{(2k-1)^2\pi^2}{2\varepsilon^2}}}{\varepsilon^2} \left[1 + \frac{\varepsilon^2}{\pi^2(2k-1)^2}\right]\end{aligned}$$

finishing the proof. ■

From the proof of proposition 2.14, exact asymptotics can be given both for $\varepsilon \to 0$ and $\varepsilon \to \infty$. Indeed:

Theorem 2.15. *The following asymptotic relations hold*

$$\mathbb{E}[N^\varepsilon] \sim \mathbb{P}(R \geq \varepsilon) \qquad \text{as } \varepsilon \to \infty$$
$$N^\varepsilon \sim \frac{1}{2\varepsilon^2} \qquad \text{a.s. as } \varepsilon \to 0$$

where the expression for $\mathbb{P}(R \geq \varepsilon)$ is known and given in the statement of proposition 2.14. Furthermore, the expectation of N^ε is continuous in ε. More precisely, for all $\lambda > 0$

$$\sum_{k \geq n+1} \mathbb{P}(S_{k-1}^\varepsilon \leq 1) \leq e^\lambda \operatorname{csch}^2(\varepsilon\sqrt{2\lambda}) \operatorname{sech}^{2(n-1)}(\varepsilon\sqrt{2\lambda}),$$

when $n = 1$, this bound implies

$$\mathbb{E}[N^\varepsilon] \leq \mathbb{P}(R \geq \varepsilon) + e^\lambda \operatorname{csch}^2(\varepsilon\sqrt{2\lambda})$$

and this bound is optimal for $\lambda > 0$ satisfying the transcendental equation

$$2\varepsilon^2 \coth^2(\varepsilon\sqrt{2\lambda}) = \lambda\,.$$

Remark 2.16. Jean Picard established the asymptotics of N^ε as $\varepsilon \to 0$ by using the self-similarity of Brownian motion to deduce the asymptotic dependence as $\varepsilon \to 0$. This approach can be extended to processes which are self-similar, but are not necessarily Markov, such as fractional Brownian motion. If a process X is self-similar, the almost sure asymptotic dependence of N^ε is given by $O(\varepsilon^{-\alpha})$ where α is the scale invariance exponent, *i.e.* such that, in distribution, $\lambda X_t = X_{\lambda^\alpha t}$. More details on this can be found in [33].

Proof. The asymptotic in the $\varepsilon \to \infty$ regime follows from an application of theorem 2.10. Similarly, an application of theorem 2.13 yields the second result since $\mathbb{E}[T^\varepsilon] = \varepsilon^2$.

Note that $T^\varepsilon = \inf\{t \geq 0 \mid |B_t| = \varepsilon\}$. For every $\mu > 0$,

$$M_t^\mu := e^{\mu B_t - \frac{\mu^2}{2} t}$$

is a bounded martingale on $[0, T^\varepsilon]$. By Doob's stopping theorem, we have $\mathbb{E}[M_t^\mu] = 1$. But B_{T^ε} and T^ε are independent from one another, so

$$\mathbb{E}\left[e^{-\frac{\mu^2}{2} T^\varepsilon}\right] = \left(\mathbb{E}\left[e^{\mu B_{T^\varepsilon}}\right]\right)^{-1} = (\cosh(\mu\varepsilon))^{-1}.$$

since $\mathbb{E}\big[f(T^\varepsilon)\mathbf{1}_{\{B_{T^\varepsilon}=\varepsilon\}}\big] = \mathbb{E}\big[f(T^\varepsilon)\mathbf{1}_{\{B_{T^\varepsilon}=-\varepsilon\}}\big] = \frac{1}{2}\mathbb{E}[f(T^\varepsilon)]$. Rewriting this, we have that for $\lambda > 0$

$$\mathbb{E}\left[e^{-\lambda T^\varepsilon}\right] = \operatorname{sech}(\varepsilon\sqrt{2\lambda})$$

Alternatively, this result is well-known and tabulated in Borodin and Salminen's book [4, p.355]. Since the increments of the S_i^ε and T_i^ε are i.i.d.,

$$\mathbb{E}\left[e^{-\lambda S_{k-1}^\varepsilon}\right] = \mathbb{E}\left[e^{-\lambda T^\varepsilon}\right]^{2(k-1)} = \operatorname{sech}^{2(k-1)}(\varepsilon\sqrt{2\lambda})$$

Using Chernoff's bound for $\lambda > 0$

$$\mathbb{P}(S_{k-1}^\varepsilon \leq 1) \leq \frac{\mathbb{E}\left[e^{-\lambda S_{k-1}^\varepsilon}\right]}{e^{-\lambda}} = e^\lambda \operatorname{sech}^{2(k-1)}(\varepsilon\sqrt{2\lambda}) \tag{2.3}$$

Fixing $\varepsilon > 0$ and using the Chernoff bound (equation 2.3), for all $\lambda > 0$

$$\begin{aligned} \sum_{k \geq n+1} \mathbb{P}(S_{k-1}^\varepsilon \leq 1) &\leq e^\lambda \sum_{k \geq n} \operatorname{sech}^{2k}(\varepsilon\sqrt{2\lambda}) \\ &= e^\lambda \operatorname{csch}^2(\varepsilon\sqrt{2\lambda}) \operatorname{sech}^{2(n-1)}(\varepsilon\sqrt{2\lambda}) \end{aligned} \tag{2.4}$$

This bound on the tail of the series shows that the series for $\mathbb{E}[N^\varepsilon]$ converges uniformly on every compact subset of $]0, \infty[$, since $\operatorname{sech}(z)$ is strictly less than 1 and decreasing for $z > 0$. Since every term in the series for $\mathbb{E}[N^\varepsilon]$ is continuous, since f_T is continuous in ε (as it can be written as the derivative of a difference of ϑ-functions), $\mathbb{E}[N^\varepsilon]$ is continuous as well.

Finally, taking $n = 1$, we get the bound for $\mathbb{E}[N^\varepsilon]$ in the statement of the theorem, which is optimal for λ satisfying the transcendental equation in the theorem. ∎

A similar result inspired from the proof of theorem 2.15 follows.

Theorem 2.17. *The function* $\mathbb{E}\left[N_B^{x,x+\varepsilon}\right]$ *is analytic for* $x \neq 0$ *and* $\varepsilon > 0$.

Proof. Following Chazal and Divol, we can write a sequence of stopping times which count the number of downcrossings between levels x and $x+\varepsilon$. The increments between these times are all identically distributed with distribution that of the hitting time of level ε by a Brownian motion (except the first of these times, which has the distribution of the first hitting time of level x by a Brownian motion). Let us denote this hitting time T^ε. By an analogous reasoning to that of the proof of theorem 2.15, we have that

$$\mathbb{P}(N^{x,x+\varepsilon} \geq k) \leq e^{\lambda}\mathbb{E}\left[e^{-\lambda T^x}\right]\mathbb{E}\left[e^{-\lambda T^\varepsilon}\right]^{2k}.$$

Doob's stopping theorem implies that for $\lambda > 0$

$$\mathbb{E}\left[e^{-\lambda T^a}\right] = e^{-a\sqrt{2\lambda}},$$

so

$$\sum_{k=n}^{\infty}\mathbb{P}(N^{x,x+\varepsilon} \geq k) \leq \frac{e^{\lambda-\sqrt{2}x}}{e^{2\varepsilon\sqrt{2\lambda}}-1}e^{-2\varepsilon(n-1)\sqrt{2\lambda}}.$$

The inequality above implies that the series converges uniformly and absolutely on every compact where $\varepsilon > 0$ and every term in the series of $\mathbb{E}[N^{x,x+\varepsilon}]$ is continuous in x and ε, so the series is continuous as well for $\varepsilon > 0$. In fact, the above inequality shows more. Indeed, the bound found above in fact gives a bound for $|\mathbb{P}(N^{x,x+\varepsilon} \geq k)|$ on an open wedge containing the strictly positive real axis in the complex plane. Indeed,

$$\mathbb{P}(N^{x,x+\varepsilon} \geq k) = \int_{\Sigma_{2k-1}} d^{2k-1}\mathbf{t}\ \psi(x,t_1)\prod_{i=2}^{2k-1}\psi(\varepsilon,t_i)$$

Let us now complexify the problem and consider a point $\varepsilon e^{i\theta}$ on an open wedge $\theta \in]-\theta_0,\theta_0[$ for some small enough $\theta_0 > 0$ and for $\varepsilon > 0$

$$\begin{aligned}\left|\prod_{i=2}^{2k-1}\psi(\varepsilon e^{i\theta},t_i)\right| &= \left|u(\mathbf{t})\,\varepsilon^{2k-2}e^{(2k-2)i\theta}e^{-\varepsilon^2 e^{2i\theta}v(\mathbf{t})}\right| \\ &= u(\mathbf{t})\,\varepsilon^{2k-2}e^{-\varepsilon^2\cos(2\theta)v(\mathbf{t})}\end{aligned}$$

where u and v are real functions of $\mathbf{t}$.

$$\begin{aligned}\left|\mathbb{P}(N^{x,x+\varepsilon e^{i\theta}} \geq k)\right| &\leq \int_{\Sigma_{2k-1}} d^{2k-1}\mathbf{t}\ \left|\psi(x,t_1)\prod_{i=2}^{2k-1}\psi(\varepsilon,t_i)\right| \\ &= \frac{1}{\cos^{k-1}(2\theta)}\mathbb{P}(N^{x,x+\varepsilon\sqrt{\cos(2\theta)}} \geq k) \\ &\leq e^{\lambda-x\sqrt{2\lambda}}\cos(2\theta_0)\left[\frac{e^{-2\varepsilon\sqrt{2\lambda\cos(2\theta_0)}}}{\cos(2\theta_0)}\right]^k\end{aligned}$$

But notice that given ε, it is always possible to find λ and θ_0 such that

$$e^{-2\varepsilon\sqrt{2\lambda\cos(2\theta_0)}} < \cos(2\theta_0)\,.$$

In fact, for $\varepsilon > 0$, θ_0 can be taken arbitrarily close to $\frac{\pi}{4}$. In particular $\mathbb{E}[N^{x,x+\varepsilon}]$ converges uniformly and absolutely on every compact subset of the open wedge $]-\theta_0, \theta_0[$ (in fact on the whole half plane). Since each term of the sum is analytic in this region, $\mathbb{E}[N^{x,x+\varepsilon}]$ is also analytic on this wedge. A similar reasoning shows the result for analyticity in x. ■

As we will later see, this theorem will imply that N^ε is analytic as well.

2.5.2 Link between $N^{x,x+\varepsilon}$ and N^ε and a notion of local time

On a tree T_f, we can define a notion of integration by defining the unique atomless Borel measure λ which is characterized by the property that every geodesic segment on T_f has measure equal to its length. Formally, we can express λ in two ways [33]

$$\lambda = \int_{\mathbb{R}} dx \sum_{\substack{\tau\in T_f\\ f(\tau)=x}} \delta_\tau \quad \text{and} \quad \lambda = \int_0^\infty d\varepsilon \sum_{\substack{\tau\in T_f\\ h(\tau)=\varepsilon}} \delta_\tau$$

By using the second way of writing λ, the identity

$$\lambda(T_f^\varepsilon) = \int_\varepsilon^\infty N^a \, da$$

is clear, as every sum in the second expression is finite for all $\varepsilon > 0$ and has N^ε terms. Of course, we could have very well have written it using the first sum, but this poses the difficulty that if T_f is infinite, so is the sum considered in this formal expression for at least some value of x. However, the restricted sum

$$\sum_{\substack{\tau\in T_f\\ f(\tau)=x\\ h(\tau)\geq\varepsilon}} \delta_\tau$$

is finite for all $\varepsilon > 0$ and there are exactly $N^{x,x+\varepsilon}$ terms in this sum. It follows that it is also possible to write

$$\lambda(T_f^\varepsilon) = \int_{\mathbb{R}} N^{x,x+\varepsilon} \, dx\,,$$

so more information is contained in $N^{x,x+\varepsilon}$ than in $\lambda(T_f^\varepsilon)$ (and by extension than in N^ε). This calculation provides the connection between Chazal and Divol and Baryshnikov's functional $N^{x,x+\varepsilon}$ and the functional detailed in this paper N^ε, since N^ε is nothing other than the derivative of $\lambda(T_f^\varepsilon)$.

For Brownian motion, $N_B^{x,x+\varepsilon}$ is asymptotic to the local time of Brownian motion at x as $\varepsilon \to 0$. This implies that

$$\begin{aligned}
\mathbb{E}[\lambda(T_B^\varepsilon)] &= \int_{\mathbb{R}} \mathbb{E}\left[N_B^{x,x+\varepsilon}\right] \, dx \\
&\sim \frac{1}{2\varepsilon} \int_{\mathbb{R}} dx \int_0^1 \varphi(x,s) \, ds \quad \text{as } \varepsilon \to 0 \\
&= \frac{1}{2\varepsilon} \,, \qquad (2.5)
\end{aligned}$$

which yields the correct asymptotics for $\lambda(T_f^\varepsilon)$ in the small ε regime. Since $\mathbb{E}\left[N_B^{x,x+\varepsilon}\right]$ is analytic in ε, this immediately shows

Proposition 2.18. The expectation $\mathbb{E}[N_B^\varepsilon]$ is analytic in ε.

Furthermore, these remarks explain the similarity between the asymptotics of the formulæ obtained by Chazal and Divol and Baryshnikov (*cf.* proposition 2.4), despite the fact that Baryshnikov considers the Brownian motion with a drift over the ray $[0, \infty[$.

This similarity can be explained in terms of the regularity of the processes. Indeed, Brownian motion with drift almost surely escapes outside of every compact set in finite time. We may thus restrict ourselves to some compact set, over which the integration over x in the above equations yields a finite result. Since B^μ is a Lévy process with stable branching process, the small scale behaviour of the Brownian motion is not affected by the addition of the drift and the limits in Picard's theorem exist, yielding exact asymptotics. Over a compact set, $\mathbb{E}[\lambda(T_{B^\mu}^\varepsilon)]$ is finite and its asymptotic behaviour as $\varepsilon \to 0$ is that of $\mathbb{E}[N^{x,x+\varepsilon}]$, so we conclude that $\mathbb{E}\left[N_{B^\mu}^{x,x+\varepsilon}\right] = O(\varepsilon^{-1})$ as $\varepsilon \to 0$ as desired. This reasoning suggests that

Conjecture 2.19. Brownian motion with any smooth drift will also have an $O(\varepsilon^{-1})$ divergence in $\mathbb{E}[N^{x,x+\varepsilon}]$.

More generally, an application of Picard's theorem proves the following fact.

Proposition 2.20. Let $\alpha > \frac{1}{2}$, then for any deterministic α-Hölder function f, the f-drifted Brownian motion $(B_t^f := B_t + f(t))$ over $[0,1]$ is such that almost surely, for every $\delta > 0$

$$\lambda(T_{B^f}^\varepsilon) = O(\varepsilon^{-1-\delta}) \quad \text{as } \varepsilon \to 0 \,.$$

Equation 2.5 provides an analogous expression for $\lambda(T_B^\varepsilon)$ of the Brownian motion on the interval $[0,t]$,

$$\mathbb{E}[\lambda(T_B^\varepsilon)] \sim \frac{t}{2\varepsilon} \quad \text{as } \varepsilon \to 0 \,,$$

which makes the divergence of the length of the trimmed tree explicit when $t \to \infty$. This fits the scaling law of Brownian motion, in the sense that in distribution the equality $N^{\varepsilon}_{B,[0,t]} = N^{\varepsilon/\sqrt{t}}_{B,[0,1]}$ holds (an equivalent expression for $\lambda(T^{\varepsilon}_{B})$ is also valid in distribution).

Finally, this whole discussion motivates a definition of a notion of local time for any process whose tree is of finite $\underline{\dim} = \overline{\dim}$, where $\underline{\dim}$ denotes the lower-box dimension.

Definition 2.21. Let X denote a process on a manifold M such that, almost surely,

$$p := \underline{\dim}\, T_X = \overline{\dim}\, T_X < \infty\,.$$

The **local time of X at x,** L^X_x is defined as:

$$L^X_x := \lim_{\varepsilon \to 0} 2\varepsilon^{p-1} N^{x,x+\varepsilon}_X\,.$$

2.5.3 The Brownian bridge

The Brownian bridge W is **not** a periodic Markov process (according to our convention), because it doesn't attain its infimum at 0. Nonetheless, the functionals of W_t which interest us are the same in distribution to the ones of the Brownian excursion [37], which is periodic in time, in the sense we have previously described.

The study of the Brownian bridge (or excursion) poses two main difficulties: the first is that these are **not** Lévy processes, which implies that we don't have readily avalaible fine asymptotics of N^{ε} as $\varepsilon \to 0$. The second difficulty is that the stopping times T^{ε}_i and S^{ε}_i are no longer identically distributed, and they are not – insofar as the author is aware – easily computable either. Nevertheless, we can still describe this process asymptotically as $\varepsilon \to 0$ and $\varepsilon \to \infty$.

These limitations remain technical and intuitively, one would like to believe that the small bars of W_t should still have the same asymptotic behaviour as in the Brownian case.

As before, the general strategy is to apply theorem 2.10. The distribution of the range of the Brownian bridge is well-known and was first published by Kennedy in 1976 [22]. Along with an application of Picard's theorem (theorem 2.12), this gives the following proposition.

Proposition 2.22. The following asymptotic relations hold

$$\mathbb{E}[N^{\varepsilon}] \sim 2\sum_{k=1}^{\infty}(4k^2\varepsilon^2 - 1)e^{-2k^2\varepsilon^2} \quad \text{as } \varepsilon \to \infty$$
$$N^{\varepsilon} = O(\varepsilon^{-2-\delta}) \quad \text{as } \varepsilon \to 0,\ \forall\, \delta > 0$$

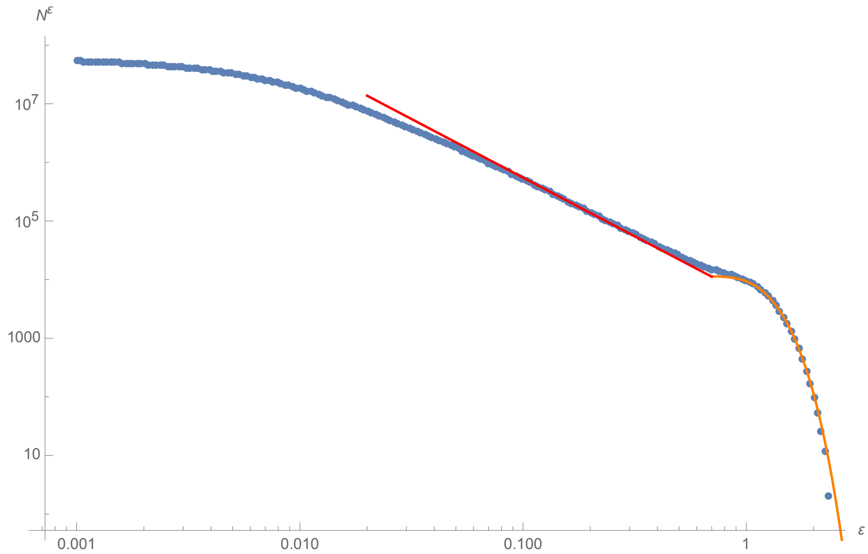

Figure 6: $\mathbb{E}[N^\varepsilon]$ as calculated by simulations of W_t, performed using a stochastic version of Euler's algorithm and the stochastic differential equation for W_t. In red, we have depicted the conjectured asymptotic result as $\varepsilon \to 0$ and in orange, the asymptotic result for $\varepsilon \to \infty$.

3 C^0-processes as limiting objects

The fact that the metric space of barcodes, equipped with the bottleneck distance, is stable with respect to the L^∞-norm is a well-established fact in persistence theory [7,31]. This stability justifies using barcodes as invariants of filtered topological spaces X. As shown in [32] and by Le Gall on $[0,1]$ [13], there is an analogous theorem of L^∞-stability for trees, where the space of trees is now equipped with the Gromov-Hausdorff distance.

An interesting point of view relative to these stability results and the results of this paper is to consider these irregular processes as almost sure C^0-limits of smooth processes, which have traditionally been more difficult to study. In this way, we can make affirmations about the barcodes of smooth processes up to some (small) error. This way of thinking is inspired by the study of trees, ultralimits and asymptotic cones in geometric group theory [35].

To illustrate the usefulness of this paradigm, we will concern ourselves in this section with the study of random trigonometric polynomials, which admit a C^0-limit with infinite tree of finite upper-box dimension. The Karhunen-Loève theorem [21,26] shows that many well-known C^0-processes can be seen as such limits and that such a reasoning would also be applicable in higher dimensions. In fact, Brownian motion itself can be seen as such a limit [25,29]. Indeed, Paul Lévy showed that if $(\xi_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. standard normal variables, the series

$$\xi_0 t + \frac{\sqrt{2}}{\pi} \sum_{k=1}^{\infty} \xi_k \frac{\sin(\pi k t)}{k}$$

almost surely uniformly converges to the standard Brownian motion on $[0,1]$. From this, we can also extract a representation for the Brownian bridge as

$$W_t = \frac{\sqrt{2}}{\pi} \sum_{k=1}^{\infty} \xi_k \frac{\sin(\pi k t)}{k} .$$

Throughout this section, we will mostly be concerned with a generalization of this series, namely

$$X_t^{\beta} := \sum_{k=1}^{\infty} \frac{\xi_k}{\pi^{\beta} k^{\beta}} \sin(\pi k t)$$

for $\frac{1}{2} < \beta < \frac{3}{2}$, which we will also show falls under our scope of assumptions.

3.1 Consequences of L^{∞}-stability

If a continuous stochastic process X (now not necessarily Markov nor 1D) has some series representation which almost surely converges uniformly, we will denote $S_n X$ the nth partial sum of this series representation of X. For any function of process Y, We will denote T_Y the tree associated to Y and N_Y^{ε} the number of bars of length $\geq \varepsilon$ of $\mathcal{B}(Y)$.

By construction, $S_n X$ is a.s. L^{∞}-close to X. It follows that the barcodes and trees of $S_n X$ and X are themselves close, with respect to the correct notions of distance. If we know the barcode of X, then up to some (small) error, we can also deduce the barcode of $S_n X$. The following theorem gives a quantitative account of this fact.

Theorem 3.1. *Let $\delta_n := \|X - S_n X\|_{L^{\infty}}$, then there exists a δ_n-matching between the barcodes of $S_n X$ and X. In particular, for any $\varepsilon \geq 2\delta_n$*

$$N_X^{\varepsilon+\delta_n} \leq N_{S_n X}^{\varepsilon} \leq N_X^{\varepsilon-\delta_n}$$

Moreover, if $\mathbb{E}[N_X^{\varepsilon}]$ is continuous with respect to ε, then

$$N_{S_n X}^{\varepsilon} \xrightarrow[n\to\infty]{L^1} N_X^{\varepsilon} \quad \text{and} \quad N_{S_n X}^{\varepsilon} \xrightarrow[n\to\infty]{\mathbb{P}} N_X^{\varepsilon} ,$$

which at fixed n quantitatively translates to

$$\mathbb{E}\left[\left|N_X^{\varepsilon} - N_{S_n X}^{\varepsilon}\right|\right] \leq 2\omega_{\varepsilon}(\delta_n) \quad \text{and} \quad \mathbb{P}(\left|N_X^{\varepsilon} - N_{S_n X}^{\varepsilon}\right| \geq k) \leq \frac{2\omega_{\varepsilon}(\delta_n)}{k}$$

where ω_{ε} is the modulus of continuity of $\mathbb{E}[N_X^{\varepsilon}]$ on the interval $[\varepsilon - \delta_n, \varepsilon + \delta_n]$. Finally, the following inequalities also hold

$$N_{S_n X}^{\delta_n} \geq N_X^{2\delta_n} \quad \text{and} \quad N_X^{\delta_n} \geq N_{S_n X}^{2\delta_n} .$$

Remark 3.2. An analogous theorem holds for $N_{S_nX}^{x,x+\varepsilon}$ and $N_X^{x,x+\varepsilon}$. In particular for $\varepsilon \geq 2\delta_n$, $N_{S_nX}^{x,x+\varepsilon}$ also converges in L^1 to $N_X^{x,x+\varepsilon}$ as soon as $\mathbb{E}\Big[N_X^{x,x+\varepsilon}\Big]$ is continuous.

Proof. The L^∞-stability of barcodes with respect to the L^∞-distance tells us that $\delta_n := \|X - S_nX\|_{L^\infty}$ controls the bottleneck distance between the two barcodes [7, 31]. This immediately implies that there exists a δ_n-matching between the barcodes of S_nX and that of X. This means that:

- If $\alpha \in \mathcal{B}(X)$ has length $|\alpha| \geq 2\delta_n$, then $\exists!\, \beta \in \mathcal{B}(S_nX)$ such that (α, β) are matched and the difference $||\alpha| - |\beta|| \leq \delta_n$;
- If $\beta \in \mathcal{B}(S_nX)$ has length $|\beta| \geq 2\delta_n$, then $\exists!\, \alpha \in \mathcal{B}(X)$ such that (α, β) are matched and the difference $||\alpha| - |\beta|| \leq \delta_n$;
- If $\alpha \in \mathcal{B}(X)$ or $\beta \in \mathcal{B}(S_nX)$ in unmatched, then they have length $\leq 2\delta_n$.

And so, the bars of length $\geq 2\delta_n$ are injectively mapped onto the δ_n-trimmed trees by the matching. By the connection between the barcodes and that of trees (algorithm 1 of [32]), we have the two following inequalities

$$\begin{aligned} N_{S_nX}^{2\delta_n} &\leq N_X^{\delta_n} \\ N_X^{2\delta_n} &\leq N_{S_nX}^{\delta_n} \end{aligned}$$

More generally and for the same reasons, for $\varepsilon \geq 2\delta_n$ we have inequalities

$$\begin{aligned} N_{S_nX}^{\varepsilon+\delta_n} &\leq N_X^{\varepsilon} \\ N_X^{\varepsilon+\delta_n} &\leq N_{S_nX}^{\varepsilon} \end{aligned}$$

We conclude from this that we have bounds on $N_{S_nX}^{\varepsilon}$

$$N_X^{\varepsilon+\delta_n} \leq N_{S_nX}^{\varepsilon} \leq N_X^{\varepsilon-\delta_n}$$

These inequalities imply

$$\left|N_{S_nX}^{\varepsilon} - N_X^{\varepsilon}\right| \leq \left|N_X^{\varepsilon+\delta_n} - N_X^{\varepsilon}\right| \vee \left|N_X^{\varepsilon-\delta_n} - N_X^{\varepsilon}\right|$$

Taking expectations of both sides, we have

$$\begin{aligned} \mathbb{E}\left[\left|N_{S_nX}^{\varepsilon} - N_X^{\varepsilon}\right|\right] &\leq \mathbb{E}\left[\left|N_X^{\varepsilon+\delta_n} - N_X^{\varepsilon}\right| \vee \left|N_X^{\varepsilon-\delta_n} - N_X^{\varepsilon}\right|\right] \\ &\leq \mathbb{E}\left[\left|N_X^{\varepsilon+\delta_n} - N_X^{\varepsilon}\right|\right] + \mathbb{E}\left[\left|N_X^{\varepsilon-\delta_n} - N_X^{\varepsilon}\right|\right] \\ &= \mathbb{E}\left[N_X^{\varepsilon} - N_X^{\varepsilon+\delta_n}\right] + \mathbb{E}\left[N_X^{\varepsilon-\delta_n} - N_X^{\varepsilon}\right] \end{aligned}$$

by monotonicity of N_X^ε. The right hand side of the inequality tends to 0 as $n \to \infty$ by continuity of $\mathbb{E}[N^\varepsilon]$, so $N_{S_nX}^\varepsilon \xrightarrow[n\to\infty]{L^1} N_X^\varepsilon$. But by Markov's inequality, L^1-convergence implies convergence in probability which finishes the proof. Spelling out Markov's inequality, we get a quantitative estimate on the difference

$$\begin{aligned}\mathbb{P}(\left|N_X^\varepsilon - N_{S_nX}^\varepsilon\right| \geq k) &\leq \frac{2(\mathbb{E}\left[N_X^\varepsilon - N_X^{\varepsilon+\delta_n}\right] \vee \mathbb{E}\left[N_X^{\varepsilon-\delta_n} - N_X^\varepsilon\right])}{k} \\ &\leq \frac{2\omega_\varepsilon(\delta_n)}{k}\end{aligned}$$

where ω_ε denotes the modulus of continuity of $\mathbb{E}[N_X^\varepsilon]$ on $[\varepsilon-\delta_n, \varepsilon+\delta_n]$. ∎

Remark 3.3. If $B = X$, we know that $\mathbb{E}[N_B^\varepsilon]$ is analytic in ε, so in particular Lipschitz, and can get good estimates for the Lipschitz constant by examining the derivative of $\mathbb{E}[N_B^\varepsilon]$.

3.1.1 Very deep critical points

We can interpret the proof of theorem 3.1 in terms of *very deep critical points*. By this, we mean critical points of S_nX which have depth at least ε, where ε does not depend on n and is taken to be some (relatively large) constant. The interpretation of theorem 3.1 in this context is that under suitable appropriate hypotheses on X, the number of very deep critical points of S_nX is constant, *i.e.* asymptotically it does not depend on n.

The statement of the previous theorem states that at fixed $\varepsilon \geq 2\delta_n$ with high probability N_X^ε and $N_{S_nX}^\varepsilon$ are roughly the same. If we suppose that N_X^ε is concentrated around some constant, then $N_{S_nX}^\varepsilon$ must also be roughly constant for some n large enough. If X is Markov, this concentration always happens in the $\varepsilon \to \infty$ regime.

Indeed, since

$$\mathbb{E}\left[(N^\varepsilon)^2\right] = \sum_{k\geq 1}(2k-1)\,\mathbb{P}(N^\varepsilon \geq k)\,,$$

the bounds on $\mathbb{P}(N^\varepsilon \geq k)$ provided by theorem 2.10 yield an inequality on the $\mathrm{Var}(N^\varepsilon)$. For non-periodic processes, this inequality is

$$\mathrm{Var}(N_X^\varepsilon) \leq \mathbb{P}(R_X \geq \varepsilon)\left[1 - \mathbb{P}(R_X \geq \varepsilon) + \frac{2\mathbb{P}(R_X \geq \varepsilon)^2}{(1-\mathbb{P}(R_X \geq \varepsilon)^2)}\right]. \tag{3.6}$$

If ε is large, $\mathbb{P}(R_X \geq \varepsilon)$ is small, so inequality 3.6 shows that with high probability, N^ε is concentrated around its mean with relatively small variation with respect to n. For n large enough, it follows that since $N_{S_nX}^\varepsilon \approx N_X^\varepsilon$, as $\varepsilon \to \infty$, $N_{S_nX}^\varepsilon$ can be taken to be constant with respect to n.

Remark 3.4. An analogous expression to equation 3.6 can also be found for Markov processes which are periodic in time. Furthermore, similar bounds can be obtained in L^r by noticing that for any $r \in \mathbb{N}^*$

$$\mathbb{E}[(N^\varepsilon)^r] = \sum_{k\geq 1} (k^r - (k-1)^r)\,\mathbb{P}(N^\varepsilon \geq k)$$

3.2 Probable and almost sure $L^\infty(E)$-rates of convergence

The previous discussion has highlighted the importance of controlling the rates of convergence of $S_nX \to X$. We will elaborate on this question and give some estimates of these rates of convergence for a large class of (non-necessarily 1D) processes. In particular, random trigonometric polynomials and their limits fall under the scope of these results.

3.2.1 Subgaussian random Fourier series

Kahane gave a bound for the L^∞-norm of finite random functions with weak hypotheses in his celebrated book [20]. In what will follow, we extend his results to infinite series. For the rest of this section, we place ourselves on a measured space E with positive finite measure μ, and denote B a linear space of measurable functions on E with values in $\mathbb{C}$, closed under complex conjugation. Recall also that

Definition 3.5. A (real) random variable X is said to be **subgaussian** if and only if

$$\mathbb{E}\left[e^{\lambda X}\right] \leq e^{\lambda^2/2}$$

In this setting,

Theorem 3.6 (Kahane, §6 [20])**.** *Let $(\xi_k)_k$ be a (finite) sequence of independent subgaussian random variables and $(f_k)_k$ be a (finite) sequence of functions of B. Let*

$$P = \sum_k \xi_k f_k$$

and suppose that there exists a measurable set I of measure $\mu(I) \geq \frac{1}{\rho}$ for some $\rho > 0$ such that for all $t \in I$

$$|P(t)| > \frac{1}{2}\,\|P\|_{L^\infty}$$

Setting $M := \|P\|_{L^\infty}$ and $r := \sum_k \|f_k\|_{L^\infty}^2$, for all $\kappa > 2$ we have

$$\mathbb{P}(M \geq 2\sqrt{2r\log(2\rho\kappa)}) \leq \frac{1}{\kappa}$$

Remark 3.7. As long as we are not concerned with finding optimal constants, we can afford to be polynomially sloppy in our estimates of $\mu(I)$, since the estimate depends on $\sqrt{\log(\rho)}$.

We can generalize this result to infinite series as follows.

Theorem 3.8. *Let $(\xi_k)_{k\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$ be infinite sequences of subgaussian random variables and functions of B respectively. Define*

$$Q := \sum_k \xi_k f_k$$

and suppose that almost surely $Q \in B$. For every $k \in \mathbb{N}$, suppose that

$$P_k := \sum_{\ell=2^{k-1}+1}^{2^k} \xi_\ell f_\ell$$

satisfies the hypotheses of Kahane's theorem with set I_k of measure $\mu(I_k) \geq \frac{1}{\rho_k}$ for $\rho_k > 0$ and define r_k in the same way as in Kahane's theorem. Then, there exists an almost surely finite constant $C > 0$ such that

$$\|Q\|_{L^\infty} \leq C \sum_{k=1}^{\infty} \sqrt{r_k \log(2^{k+1}\rho_k)}$$

Proof. If the series $\sum_k \sqrt{r_k \log(2^{k+1}\rho_k)}$ diverges, the statement of the theorem is vacuous, so let us suppose the series converges. Denote E_k the event defined by

$$E_k := \{\|P_k\|_{L^\infty} \geq 2\sqrt{2r_k \log(2^{k+1}\rho_k)}\}$$

By Kahane's theorem, this event has probability $\mathbb{P}(E_k) \leq 2^{-k}$. The Borel-Cantelli lemma implies that only finitely many E_k can happen. In other words, there exists a random variable N such that almost surely for all $k \geq N$ we have

$$\|P_k\|_{L^\infty} \leq 2\sqrt{2r_k \log(2^{k+1}\rho_k)}$$

It follows that:

$$\begin{aligned}
\|Q\|_{L^\infty} &\leq \left\| \sum_{k=1}^{N-1} P_k \right\| + \sum_{k=N}^{\infty} 2\sqrt{2r_k \log(2^{k+1}\rho_k)} \\
&\leq C \sum_{k=1}^{\infty} \sqrt{r_k \log(2^{k+1}\rho_k)}
\end{aligned}$$

For some almost surely finite C, since the first sum is almost surely finite and so Kahane's theorem ensures that the sum has almost sure L^∞-norm. ∎

Remark 3.9. If $B = C^0$, we can check the condition that Q is almost surely in B by virtue of Kolmogorov's criterion.

This last corollary immediately gives bounds on the L^∞-rates of convergence of random series which have subgaussian coefficients by setting Q to be the tail of the random series. The almost surely finite constant of theorem 3.8 can be estimated as a *probable* effective constant, as we will show in the following section.

3.2.2 A quantitative Borel-Cantelli lemma

Given a sequence of events $(E_k)_{k\in\mathbb{N}}$ the Borel-Cantelli lemma states that

$$\sum_{k=1}^{\infty} \mathbb{P}(E_k) \;\Rightarrow\; \mathbb{P}\left[\limsup_{k\to\infty} E_k\right] = 0\,.$$

In what will follow, it can be helpful to take interest in the rate of convergence of the limit in the lemma of the theorem. This will allows us to give bounds for the probability of the events $\bigcup_{k\geq n} E_k$. The proof of Borel-Cantelli gives a hint on the rate of convergence of these probabilities. First, note that:

$$\bigcup_{k\geq 1} E_k \supseteq \bigcup_{k\geq 2} E_k \supseteq \bigcup_{k\geq 3} E_k \supseteq \cdots \supseteq \limsup_{k\to\infty} E_k$$

The statement of the lemma can then be translated as follows.

Lemma 3.10 (Quantitative Borel-Cantelli)**.** The random variable N defined for $\omega \in \Omega$ by:

$$N(\omega) = \inf_{k\in\mathbb{N}} \left\{ \omega \notin \bigcup_{p\geq k} E_p \right\}$$

is almost surely finite. Furthermore,

$$\mathbb{P}\left[\bigcup_{k\geq m} E_k\right] \leq \sum_{k\geq m} \mathbb{P}(E_k)\,.$$

Proof. The event $\{N \geq m\}$ is in fact

$$\{N \geq m\} = \bigcap_{n=m}^{\infty} \bigcup_{k\geq n} E_k = \bigcup_{k\geq m} E_k$$

and the measure of this event is bounded by

$$\mathbb{P}(N \geq m) = \mathbb{P}\left[\bigcup_{k\geq m} E_k\right] \leq \sum_{k\geq m} \mathbb{P}(E_k)$$

which guarantees the almost sure finiteness of N as soon as $\sum_k \mathbb{P}(E_k)$ converges. ∎

Applying this logic to the almost surely finite constant C of theorem 3.8, we see a bound on C can be given provided that we control the number of terms in the finite series and give a statement about the size of this finite sum. That is, we need to ensure that the random variable $N < m$ for some $m \in \mathbb{N}$ and that

$$\|R_m\| := \left\| \sum_{k=1}^{m} P_k \right\|_{L^\infty} < K$$

for some constant K. So, a bound on C depending on m and K holds if the event $\{R_m < K\} \cap \{N < m\}$ occurs. This happens with probability

$$\mathbb{P}(\{R_m < K\} \cap \{N < m\}) \geq 1 - \mathbb{P}(\|R_m\| \geq K) - \mathbb{P}(N \geq m)$$

Kahane's theorem gives us a bound for $\mathbb{P}(\|R_m\| \geq K)$ and the discussion on the quantitative Borel-Cantelli lemma gives a bound on $\mathbb{P}(N \geq m)$. In this way, we have given an effective *probable* estimate of C valid with probability at least $1 - \mathbb{P}(\|R_m\| \geq K) - \mathbb{P}(N \geq m)$.

3.3 Application to random trigonometric polynomials

These results can be applied to random trigonometric polynomials. Given our previous discussion, quantifying the rates of convergence amounts to finding the Lebesgue measure of the sets I of Kahane's theorem. This has been done by Kahane in his book on random series [20], but in more refinement by Bernstein, who found that for a trigonometric polynomial of degree n, this set I can be taken to be of Lebesgue measure at least $\frac{1}{n}$ [27]. Theorem 3.8 immediately gives the rates of convergence of the series X_t^β.

Proposition 3.11. The series X^β almost surely converges in $L^\infty([0,1])$ at a rate $O(n^{\frac{1}{2}-\beta}\sqrt{\log n})$. In particular, the series representation of Brownian motion almost surely converges at a rate $O(n^{-\frac{1}{2}}\sqrt{\log n})$ and this result is optimal.

This optimality of this result is due to a result by Kühn and Linde, who showed that the *expected* optimal rate of convergence for the Brownian series is exactly of this order [24].

The in detail calculation of $\mathcal{B}(X^\beta)$ is beyond the scope of this paper, but with our results, we already have an asymptotic bound on $N^\varepsilon_{X^\beta}$ as $\varepsilon \to 0$. Using Kolmogorov's criterion, it is not hard to show that X_t^β is almost surely $(\beta - \frac{1}{2} - \delta)$-Hölder for all $\delta > 0$. Applying Picard's theorem,

$$p := \overline{\dim}\, T_{X^\beta} = \limsup_{\varepsilon \to 0} \frac{\log N^\varepsilon}{\log(1/\varepsilon)} \vee 1 = \frac{2}{2\beta - 1}\,.$$

In particular, almost surely, $N^\varepsilon_{X^\beta} = O(\varepsilon^{-p-\delta})$ for all $\delta > 0$ as $\varepsilon \to 0$. Thus, by theorem 3.1,

Proposition 3.12. For every $\varepsilon \geq 2\delta_n$ and all $\delta > 0$, the following inequality holds almost surely

$$N^{\varepsilon}_{S_n X^{\beta}} \leq O((\varepsilon - \delta_n)^{-p-\delta}) \,.$$

Furthermore,

$$\mathbb{E}\Big[\Big|N^{\varepsilon}_{S_n X^{\beta}} - N^{\varepsilon}_{X^{\beta}}\Big|\Big] \leq \mathbb{E}\Big[N^{\varepsilon-\delta_n}_{X^{\beta}} - N^{\varepsilon+\delta_n}_{X^{\beta}}\Big]$$

For the Fourier series representation of Brownian motion, an application of theorem 3.1 shows that

Proposition 3.13. For $\varepsilon \geq 2\delta_n$,

$$\mathbb{E}\big[\big|N^{\varepsilon}_{S_n B} - N^{\varepsilon}_{B}\big|\big] \leq 2\Lambda_B(\varepsilon')\delta_n \quad \text{and} \quad \mathbb{P}(\big|N^{\varepsilon}_{S_n B} - N^{\varepsilon}_{B}\big| \geq k) \leq \frac{2\Lambda_B(\varepsilon')\delta_n}{k}$$

where $\Lambda_B(\varepsilon)$ is the absolute value of the derivative of $\mathbb{E}[N^{\varepsilon}_{B}]$ with respect to ε and $\varepsilon' \in [\varepsilon - \delta_n, \varepsilon + \delta_n]$. Furthermore, almost surely, for $\varepsilon \geq 2\delta_n$ small enough,

$$\big|N^{\varepsilon}_{S_n B} - N^{\varepsilon}_{B}\big| \lesssim \frac{2\delta_n \varepsilon}{(\varepsilon^2 - \delta_n^2)^2} = O\left(\frac{1}{\varepsilon^3}\sqrt{\frac{\log n}{n}}\right) \qquad \text{as } n \to \infty$$

Remark 3.14. Numerically, the inequalities in expectation give sharp bounds on these quantities when ε is large, as in this region Λ_B is small. For very large n and at fixed ε small enough (but far away enough from δ_n), the last formula also gives good approximations.

A similar almost sure result to that of proposition 3.13 can be obtained for the fractional Brownian motion. Indeed, if B^H denote a fractional Brownian motion of Hurst parameter H, it is well-known that B^H is a self-similar process so exact asymptotics for $N^{\varepsilon}_{B^H}$ ensue in the $\varepsilon \to 0$ regime in accordance to Picard's results [33]. Almost surely,

$$N^{\varepsilon}_{B^H} \sim \frac{c_H}{\varepsilon^{1/H}} \quad \text{as } \varepsilon \to 0 \,.$$

Fractional Brownian motion also admits a representation in the form of a Fourier series. These representations are rate-optimal of optimal for $H > \frac{1}{2}$ [17] in the sense of Kühn and Linde's results [24] and are of the form

$$B^H_t = a^H_0 \xi_0 t + \sum_{k \geq 1} a^H_k \left[\xi_k \left(\cos(\pi k t) - 1\right) + \xi'_k \sin(\pi k t)\right]$$

where $\sum_{k\geq 0}(a^H_k)^2 < \infty$ and (ξ_k) and (ξ'_k) are sequences of i.i.d standard Gaussian random variables. The precise expression of the a^H_k as well as the proof of the validity of this expansion is given in [17] for $H \leq \frac{1}{2}$ and in [18] for $H > \frac{1}{2}$. In particular, theorem 3.8 yields an estimate of the almost sure L^{∞}-rate of convergence of these series representation. Let us denote the exact rate of convergence δ_n and this almost sure estimate $\widehat{\delta_n}$. The following result follows.

Proposition 3.15. Almost surely, for $\varepsilon \geq 2\delta_n$ small enough,

$$\begin{aligned}\left|N^{\varepsilon}_{S_n B^H} - N^{\varepsilon}_{B^H}\right| &\lesssim c_H\left((\varepsilon-\delta_n)^{-\frac{1}{H}} - (\varepsilon+\delta_n)^{-\frac{1}{H}}\right) \quad \text{as } n\to\infty \\ &\leq c_H\left((\varepsilon-\widehat{\delta}_n)^{-\frac{1}{H}} - (\varepsilon+\widehat{\delta}_n)^{-\frac{1}{H}}\right) \\ &\leq \frac{2c_H\widehat{\delta}_n}{H}(\varepsilon-\widehat{\delta}_n)^{-1-\frac{1}{H}}\end{aligned}$$

3.4 Barcodes of random walks and empirical processes

The central limit theorem guarantees that a large category of processes will tend in distribution to the Brownian motion or the Brownian bridge, which further motivates our explicit calculations for these two processes. An example of sequences tending to these limits are random walks and empirical processes.

Definition 3.16. Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of (reduced, centered) i.i.d random variables. A **random walk** is the process defined by the partial sums of this sequence

$$S_n := \sum_{k=1}^{n} X_k \,.$$

The **empirical process** defined by X is the process

$$\alpha_n^X(t) := \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{]-\infty,x]}(X_k)\,.$$

Donsker's theorem guarantees the convergence in distribution of these two processes to the Brownian motion and the Brownian bridge respectively. *A priori*, this result alone is not sufficient to guarantee a statement on the barcodes. However, if we allow ourselves to make slight restrictions on the random walks considered, we can guarantee almost sure L^∞-convergence of the processes at a known rate, which by theorem 3.1 guarantees statements on their barcodes. The bound on the rate of convergence for the random walk is given by the celebrated Komlós-Major-Tusnády (KMT) theorem.

Theorem 3.17 (KMT, [23])**.** *Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of centered, reduced i.i.d. random variables such that the moment generating function of X_1 is defined on some non-empty neighborhood of 0. Then, there exists a Brownian motion $(B_t)_{t\geq 0}$ such that for all $x\in\mathbb{R}$ and all $n\in\mathbb{N}^*$, we have*

$$\mathbb{P}\left(\max_{1\leq k\leq n}|S_k - B_k| > C\log(n) + x\right) \leq Ke^{-\lambda x}$$

for some positive constants C, K and λ.

The Borel-Cantelli lemma guarantees that, almost surely,

$$\max_{1\le k\le n} |S_k - B_k| = O(\log(n))\,.$$

We can extend the random walk to $[0,n]$ by linear interpolation. With an appropriate rescaling and renormalization of the argument, it is possible to define a process on $[0,1]$ *à la* Donsker

$$T_t^{(n)} := \frac{1}{\sqrt{n}}\left(\sum_{k=1}^{\lfloor nt\rfloor} X_k + (nt - \lfloor nt\rfloor) X_{\lfloor nt\rfloor+1}\right)\,.$$

For this new extended process T_t, the KMT theorem implies a bound on the L^∞-norm to a Brownian motion.

Proposition 3.18. There exists a Brownian motion $(B_t)_{t\ge 0}$ such that almost surely, for n large enough,

$$\delta_n := \left\| B_t - T_t^{(n)} \right\|_{L^\infty([0,1])} = O(n^{-\frac{1}{2}}\log n)\,.$$

Proof. For $h := \frac{1}{n}$ small enough, it is a classical result by Lévy [29, Theorem 1.14] that the almost sure modulus of continuity of Brownian motion is

$$\omega(h) = O(\sqrt{h\log(1/h)})\,.$$

The estimation of the corollary follows, as this modulus of continuity controls the fluctuations of Brownian motion on every interval of size $\frac{1}{n}$. ■

With this estimation on the rate of convergence, the bounds on $N^\varepsilon_{T^{(n)}}$ can be rendered precise by applying theorem 3.1.

Proposition 3.19. For $\varepsilon \ge 2\delta_n$,

$$\mathbb{E}\left[\left|N^\varepsilon_{T^{(n)}} - N^\varepsilon_B\right|\right] \le 2\Lambda_B(\varepsilon')\delta_n \quad \text{and} \quad \mathbb{P}(\left|N^\varepsilon_{T^{(n)}} - N^\varepsilon_B\right| \ge k) \le \frac{2\Lambda_B(\varepsilon')\delta_n}{k}\,,$$

and almost surely, for $\varepsilon \ge 2\delta_n$ small enough,

$$\left|N^\varepsilon_{T^{(n)}} - N^\varepsilon_B\right| \lesssim \frac{2\delta_n\varepsilon}{(\varepsilon^2-\delta_n^2)^2} = O\left(\frac{1}{\varepsilon^3}\frac{\log n}{\sqrt{n}}\right) \quad \text{as } n\to\infty\,.$$

This last proposition fits nicely with the numerical results obtained in figure 5, which was made with $n = 5000$. We can see that at $n^{-\frac{1}{2}}\log n \sim 0.05$, the plot of $N^\varepsilon_{T^{(n)}}$ starts to diverge off of the line $\frac{1}{2\varepsilon^2}$ in the figure.

Finally, it is also possible to estimate the almost sure L^∞-rate of convergence of the empirical process to the Brownian bridge by using an alternate version of the KMT theorem.

Theorem 3.20 (KMT', [23])**.** *Let $(U_n)_{n\in\mathbb{N}^*}$ be an i.i.d. sequence of uniform random variables on $[0,1]$. Then, there exists a Brownian bridge $(W_t)_{1\geq t\geq 0}$ such that for all $n \in \mathbb{N}^*$ and all $x > 0$*

$$\mathbb{P}\Big(\big\|\alpha_n^U(t) - W_t\big\|_{L^\infty} > n^{-\frac{1}{2}}(C\log(n) + x)\Big) \leq Le^{-\lambda x}$$

for some universal positive constants C, L and λ which are explicitly known [5].

This yields, in accordance to theorem 3.1 a completely analogous statement about the barcode of the empirical process to that of proposition 3.19.

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References

[1] R. J. Adler and J. E. Taylor. *Random Fields and Geometry.* Springer New York, 2007.

[2] J.-M. Azaïs and M. Wschebor. *Level Sets and Extrema of Random Processes and Fields.* John Wiley & Sons, Inc., Jul 2008.

[3] Y. Baryshnikov. Time series, persistent homology and chirality. *arXiv:1909.09846*, 2019.

[4] A. N. Borodin and P. Salminen. *Handbook of Brownian Motion - Facts and Formulae.* Birkhäuser Basel, 2002.

[5] J. Bretagnolle and P. Massart. Hungarian constructions from the nonasymptotic viewpoint. *The Annals of Probability*, 17(1):239–256, jan 1989.

[6] M. Carriere, S. Oudot, and M. Ovsjanikov. Sliced Wasserstein Kernel for Persistence Diagrams. In *ICML 2017 - Thirty-fourth International Conference on Machine Learning*, pages 1–10, Sydney, Australia, Aug. 2017.

[7] F. Chazal, V. de Silva, M. Glisse, and S. Oudot. *The Structure and Stability of Persistence Modules.* Springer International Publishing, 2016.

[8] F. Chazal and V. Divol. The density of expected persistence diagrams and its kernel based estimation. In B. Speckmann and C. D. Tóth, editors, *34th International Symposium on Computational Geometry (SoCG 2018)*, volume 99 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 26:1–26:15, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[9] D. Cohen-Steiner, H. Edelsbrunner, J. Harer, and Y. Mileyko. Lipschitz functions have L^p-stable persistence. *Foundations of Computational Mathematics*, 10(2):127–139, Jan 2010.

[10] N. Curien, J.-F. Le Gall, and G. Miermont. The Brownian cactus I. scaling limits of discrete cactuses. *Ann. Inst. H. Poincaré Probab. Statist.*, 49(2):340–373, 05 2013.

[11] J. Curry. The fiber of the persistence map for functions on the interval. *Journal of Applied and Computational Topology*, 2(3-4):301–321, dec 2018.

[12] V. Divol and T. Lacombe. Understanding the topology and the geometry of the persistence diagram space via optimal partial transport. *CoRR*, abs/1901.03048, 2019.

[13] T. Duquesne and J.-F. Le Gall. *Random trees, Lévy processes and spatial branching processes.* Number 281 in Astérisque. Société mathématique de France, 2002.

[14] T. Duquesne and J.-F. Le Gall. Probabilistic and fractal aspects of Lévy trees. *Probability Theory and Related Fields*, 131(4):553–603, Nov 2004.

[15] S. N. Evans. *Probability and Real Trees.* Springer Berlin Heidelberg, 2008.

[16] W. Feller. The asymptotic distribution of the range of sums of independent random variables. *The Annals of Mathematical Statistics*, 22(3):427–432, 1951.

[17] E. Iglói. A rate-optimal trigonometric series expansion of the fractional Brownian motion. *Electronic Journal of Probability*, 10(0):1381–1397, 2005.

[18] J. Istas. Spherical and hyperbolic fractional Brownian motion. *Electronic Communications in Probability*, 10(0):254–262, 2005.

[19] K. Itô and H. P. McKean. Diffusion processes and their sample paths. *Classics in Mathematics*, 1996.

[20] J.-P. Kahane. *Some random series of functions.* Cambridge University Press, Feb 1986.

[21] K. Karhunen. über lineare Methoden in der Wahrscheinlichkeitsrechnung. Number 37. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys., 1947.

[22] D. P. Kennedy. The distribution of the maximum Brownian excursion. *Journal of Applied Probability*, 13(2):371–376, 1976.

[23] J. Komlós, P. Major, and G. Tusnády. An approximation of partial sums of independent RV'-s, and the sample DF. i. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 32(1-2):111–131, 1975.

[24] T. Kühn and W. Linde. Optimal series representation of fractional Brownian sheets. *Bernoulli*, 8(5):669–696, 2002.

[25] J.-F. Le Gall. *Brownian Motion, Martingales, and Stochastic Calculus.* Springer International Publishing, 2016.

[26] M. Loève. *Probability Theory II*, volume 46 of *Graduate Studies in Mathematics.* Springer-Verlag New York, 4th edition, 1978.

[27] M. A. Malik and M. C. Vong. Inequalities concerning the derivative of polynomials. *Rendiconti del Circolo Matematico di Palermo*, 34(3):422–426, Oct 1985.

[28] Y. Mileyko, S. Mukherjee, and J. Harer. Probability measures on the space of persistence diagrams. *Inverse Problems*, 27(12):124007, Nov 2011.

[29] P. Mörters and Y. Peres. *Brownian Motion.* Cambridge University Press, Jan 2001.

[30] J. Neveu and J. Pitman. Renewal property of the extrema and tree property of the excursion of a one-dimensional Brownian motion. *Séminaire de probabilités de Strasbourg*, 23:239–247, 1989.

[31] S. Y. Oudot. *Persistence Theory - From Quiver Representations to Data Analysis*, volume 209 of *Mathematical surveys and monographs.* American Mathematical Society, 2015.

[32] D. Perez. On c^0-persistent homology and trees. arXiv, 2020.

[33] J. Picard. A tree approach to p-variation and to integration. *The Annals of Probability*, 36(6):2235–2279, Nov 2008.

[34] S. O. Rice. Mathematical analysis of random noise. *The Bell System Technical Journal*, 23(3):282–332, 1944.

[35] J. Roe. *Lectures on Coarse Geometry*, volume 31 of *University Lecture Series*. American Mathematical Society (AMS), 2003.

[36] K. Turner, Y. Mileyko, S. Mukherjee, and J. Harer. Fréchet means for distributions of persistence diagrams. *Discrete & Computational Geometry*, 52(1):44–70, Jul 2014.

[37] W. Vervaat. A relation between Brownian bridge and Brownian excursion. *The Annals of Probability*, 7(1):143–149, Feb 1979.